Nombres complexes

Exercice 9 Guadeloupe – Guyane – Martinique Septembre 2002 Série S – Correction

1-

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}} = z_A$$

$$\left|-\frac{1}{2}+\frac{1}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2}$$

$$-\frac{1}{2}+\frac{1}{2}i = \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$

$$-\frac{1}{2}+\frac{1}{2}i = \frac{\sqrt{2}}{2}e^{i\left(\pi-\frac{\pi}{4}\right)}$$

$$-\frac{1}{2}+\frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

$$-\frac{1}{2}+\frac{1}{2}i = \frac{\sqrt{2}}{2}e^{i\frac{3\pi}{4}} = z_B$$

2- a) $\forall\alpha\in IR, e^{i2\alpha}-1-2ie^{i\alpha}$

$$= e^{i\alpha}\left(e^{i\alpha}-e^{-i\alpha}\right)-2ie^{i\alpha}\sin\alpha$$
$$= e^{i\alpha}\left(2i\sin\alpha\right)-2ie^{i\alpha}\sin\alpha$$
$$= 0$$

Donc, $\forall\alpha\in IR, e^{i2\alpha}-1 = 2ie^{i\alpha}\sin\alpha$

b) $f(M) = MA\times MB$

$$f(M) = \left\|\overrightarrow{MA}\right\|\times\left\|\overrightarrow{MB}\right\|$$

$$f(M) = \left|1+i-e^{i\alpha}\right|\times\left|-\frac{1}{2}+\frac{1}{2}i-e^{i\alpha}\right|$$

$$f(M) = \left|\left(1+i-e^{i\alpha}\right)\left(-\frac{1}{2}+\frac{1}{2}i-e^{i\alpha}\right)\right|$$

$$f(M) = \left|-\frac{1}{2}+\frac{1}{2}i-e^{i\alpha}-\frac{1}{2}-\frac{1}{2}i-ie^{i\alpha}+\frac{1}{2}e^{i\alpha}-\frac{1}{2}ie^{i\alpha}+e^{i2\alpha}\right|$$

$$f(M) = \left|-1-\frac{1}{2}e^{i\alpha}-\frac{3}{2}ie^{i\alpha}+e^{i2\alpha}\right|$$

$$f(M) = \left|e^{i2\alpha}-1-\left(\frac{1}{2}+\frac{3}{2}i\right)e^{i\alpha}\right|$$

c) $f(M) = \left| e^{i2\alpha} - 1 - \left(\frac{1}{2} + \frac{3}{2}i \right) e^{i\alpha} \right|$

$f(M) = \left| 2ie^{i\alpha} \sin\alpha - \left(\frac{1}{2} + \frac{3}{2}i \right) e^{i\alpha} \right|$

$f(M) = \left| e^{i\alpha} \left(2i\sin\alpha - \left(\frac{1}{2} + \frac{3}{2}i \right) \right) \right|$

$f(M) = \left| e^{i\alpha} \right| \times \left| 2i\sin\alpha - \left(\frac{1}{2} + \frac{3}{2}i \right) \right|$

$f(M) = \left| 2i\sin\alpha - \left(\frac{1}{2} + \frac{3}{2}i \right) \right|$

$f(M) = \left| -\frac{1}{2} + i\left(2\sin\alpha - \frac{3}{2} \right) \right|$

$f(M) = \sqrt{\left(-\frac{1}{2} \right)^2 + \left(2\sin\alpha - \frac{3}{2} \right)^2}$

$f(M) = \sqrt{\frac{1}{4} + \left(2\sin\alpha - \frac{3}{2} \right)^2}$

3- a) $f(M)$ minimal

$\Leftrightarrow \left(-\frac{3}{2} + 2\sin\alpha \right)^2$ minimal

$\Leftrightarrow -\frac{3}{2} + 2\sin\alpha = 0$

$\Leftrightarrow \sin\alpha = \frac{3}{4}$

Rappel : $\forall x \in IR, \cos^2 x + \sin^2 x = 1$. On a donc :

$\cos\alpha = \pm\sqrt{1 - \frac{9}{16}}$

$\cos\alpha = \pm\frac{\sqrt{7}}{4}$

Il existe donc deux points de C pour lesquels $f(M)$ est minimal : $M\left(\frac{\sqrt{7}}{4} ; \frac{3}{4} \right)$ et $M\left(-\frac{\sqrt{7}}{4} ; \frac{3}{4} \right)$.

3- b) $f(M)$ maximal

$\Leftrightarrow \left(-\frac{3}{2} + 2\sin\alpha \right)^2$ maximal

$\Leftrightarrow \sin\alpha = -1$

Il existe donc un unique point de C pour lequel $f(M)$ est maximal : $M(0;-1)$.

On a alors : $f(M) = \sqrt{\frac{1}{4} + \left(-\frac{3}{2} - 2 \right)^2}$

$$f(M)=\sqrt{\frac{1}{4}+\left(-\frac{7}{2}\right)^2}$$

$$f(M)=\sqrt{\frac{1}{4}+\frac{49}{4}}$$

$$f(M)=\sqrt{\frac{50}{4}}$$

$$f(M)=\frac{5\sqrt{2}}{2}$$